900th *Birth Anniversary of Bhaskara-II* (*Born* 1114 AD):

Method of Solving an Indeterminate Equation (कुट्टक)
from *Lilavati of Bhaskara*-II.

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Introduction: - Indian Mathematicians were basically astronomers, and they stated a set of steps to solve Problems of the type $y = \frac{Ax + C}{B}$ in which A**,** B and C are known integers, and *x* & *y* are unknown integral values required. Method for solving such equations is "*Kuttaka*" or "*Vallika Kuttakara*". Problems on *Linear Indeterminate Equations* are in several ancient texts on Indian Astronomy.

Aryabhatiya of Aryabhata-I (5th c. AD) *gave* a general rule for solving *linear indeterminate equation. Bhascara I* (6thc.AD), *Brahmagupta* (8thcenturyAD), *Mahaveeracarya* (9th century AD) and others explained general methods to solve equations for integral solutions. **.**

I. Kuttaka (कुट्टक) from *Lilavati* of *Bhascara-II, a section of Siddhanta-siromani*:

In an ***i****ndeterminate linear equation* of the form $y = \frac{Ax + C}{B}$, A is dividend (भाज्य), B is divisor (भाजक) and C is addend (क्षेपक) are known integers, and *x* a multiplier (गुणक) and *y* a quotient (फलम् / लब्धि) are to be found.

Lilavati states a method to find *the least integral values* for *x* the multiplier (गुणक) and *y* the quotient (लब्धि) of the equation $y = \frac{Ax + C}{B}$ in *five steps*.

Step 1: - भाज्यो हारः क्षेपकश्चापवर्त्यः केनाप्यादौ संभवे कुट्टकार्थम् ।
येन च्छिन्नो भाज्यहारौ न तेन क्षेपश्चतद् दुष्टमुद्दिष्टमेव ॥248॥

[Ref. 1 (p.102)]

Purport: - HCF of A and B must divide C for solving the equation. Otherwise the problem is improper *(both x and y* cannot be *non-zero integers).*

Step 2: - परस्परं भाजितयोर्ययोर्यः शेषस्तयोः स्यादपवर्तनं सः ।
तेनापवर्त्तेन विभाजितौ यौ तौ भाज्यहारौ दृढसंज्ञकौ स्तः ||249||

[Ref. 1 (p.103)]

Purport: - Divide A and B mutually and find their HCF (k). Divide A, B and C by the HCF (k) and get *reduced dividend* (a); reduced *divisor* (b); and *reduced addend* (c), and get the reduced equation in the form $y = \frac{ax+c}{b} = \left[\frac{\text{A}x+\text{C}}{\text{B}}\right]$.

<u>Step 3</u>: - मिथोभजेत्तौ दृढभाज्यहारौ यावद् विभाज्ये भवतीय रूपम् ।
फलान्यधोधस्तदधो निवेश्य क्षेपस्ततः शून्यमुपान्तितेन ॥250॥
[Ref. 1 (p.103)]
Purport: - Divide the *Reduced dividend* (a) and *reduced divisor* (b) mutually (*as in the case of finding their* HCF) until the *remainder* 1 is obtained. Place the *quotients* so obtained one below the other and below them the *reduced addend* (c) and then zero.
Mathematically: *Mutual division* of dividend (a) and divisor (b) is;

$a = q_1\, b + r_1\, ;\;\; 0 \le r_1 < \; |b|,$

$b = q_2\; a + r_2\, ;\;\; 0 \le r_2 < \; r_1$; and so on, until the remainder $r_n = 1$ is obtained.

In general: Mutual division of a and b would be of the form;

$r_{(t-2)} = q_t\, r_{(t-1)} + r_t$; $0 \le r_t < r_{(t-1)}$, in which $r_{(-1)} = a$ and $r_0 = |b|$ for t = 1, 2, 3, . . . n.
When $r_n = 1$ is obtained, write the quotients $q_1, q_2, q_3, \ldots q_n$ one below the other, and then the *reduced addend* (c) and then zero.

<u>Step 4</u>: - स्वोर्ध्वे हतेऽन्त्येन युते तदन्त्यं त्यजेन्मुहुः स्यादिति राशियुग्मम् ||251(*a*)||
ऊर्ध्वो विभाज्येन दृढेन तष्टः फलं गुणो स्यादधरो हरेण ॥251(*b*)||
[Ref. 1 (p.103)]
Purport ||251(*a*)||: - Multiply the last quotient by the term below the quotient and, to the product add the last term, and write the sum against the quotient (*in the next column*). Reject the last term and repeat the process till a pair of results from the top is obtained.

॥251(*b*)||The first term (from the top thus obtained) is to be *abraded* by the *reduced dividend* to get the *remainder* (as the *quotient* y_0, लब्धि) and, 2nd term from the top thus obtained is to be *abraded* by the *reduced divisor* to get the *remainder* (as the *multiplier* x_0, गुणक).
Mathematically; ॥251(*a*)||Multiply the last quotient [$q_{(n)}$] by the term below the quotient (addend c), and to the product add the term below it (0) to get F_1 [*i.e.,*; $F_1 = (q_n)\, c + 0$], and write the sum [F_1] in the next column against the quotient [q_n].

Multiply the quotient [$q_{(n-1)}$] by the addend c, and to the product add the term below it (0) to get F_2 [*i.e.,*; $F_1 = (q_{(n-1)}\, c + 0]$, and write the sum [$F_2$] in the next column against the quotient [$q_{(n-1)}$].

$$F_{(n-t+1)} = q_t F_{(n-t)} + F_{(n-t-1)};$$
In which t = 1,2,3 . . . (n-1), n

q_t									$F_{(n-t+1)}$
q_1	q_1	q_1	q_1				q_1	q_1	F_n
q_2	q_2	q_2	q_2				q_2	$F_{(n-1)}$	$F_{(n-1)}$
q_3	q_3	q_3	q_3				$F_{(n-2)}$		$F_{(n-2)}$
.	.	.	.						
.	.	.	.		.				
.	.	.	.		.				
$q_{(n-3)}$	$q_{(n-3)}$	$q_{(n-3)}$	$q_{(n-3)}$	F_4	.				F_4
$q_{(n-2)}$	$q_{(n-2)}$	$q_{(n-2)}$	F_3						F_3
$q_{(n-1)}$	$q_{(n-1)}$	F_2							F_2
q_n	F_1								F_1
c									
0									

Mathematically: Procedure stated in the *last lines of* step 3 could be modified thus;

Modified Vallika process (||251(*b*)||) : - Write the quotients $q_1, q_2, q_3 \ldots q_n$ one below the other and below them the *reduced addend* (c), and zero.

Multiply the quotient [q_n] by the addend (c), and to the product add 0, then write the sum F_1 [= [q_n c + 0] in the left side of quotient [q_n] in the column [$F_{(n-t+1)}$].

Similarly get F_2 [*i.e.,*; $F_2 = q_{(n-1)} F_1 + c$], and write [F_2] in the left side of quotient [$q_{(n-1)}$] in the column [$F_{(n-t+1)}$].

Repeat the process from the bottom until a pair of results [F_n] and [$F_{(n-1)}$] in the top of the column [$F_{(n-t+1)}$] is obtained. This process has the name वल्लिका कुट्टकार (creeper-like process), which could be shown as a *recurring relation* thus;

$F_{(n-t+1)} = q_t F_{(n-t)} + F_{(n-t-1)}$ where $F_0 = c$ and $F_{(-1)} = 0$ for [t = n, (n-1), . . . , 2, 1].

This process is to be continued until a pair of results $F_{(n)}$ and $F_{(n-1)}$ from the top in the column [$F_{(n-t+1)}$] are obtained.

$F_{(n-t+1)} = q_t F_{(n-t)} + F_{(n-t-1)}$ *for* t = 1,2,3 . . . (n-1), n

q_t	$F_{(n-t+1)}$
q_1	F_n
q_2	$F_{(n-1)}$
.	.
.	.
.	.
$q_{(n-3)}$	F_4
$q_{(n-2)}$	F_3
$q_{(n-1)}$	F_2
q_n	F_1
c	
0	

$F_{(n-1)}$ is to be *divided* by the *reduced dividend* (a) to get the *remainder* [$F_{(n-1)} = at + y_0$]; [y_0 = लब्धि]
$F_{(n-2)}$ is to be *divided* by the *reduced divisor* (b) to get the *remainder* [$F_{(n-2)} = bt + x_0$]; [x_0 = गुणक]

Step 5: - एवं तदैवात्र यदा समास्ताः स्युर्लब्धयश्चेद्विषमस्तदानीम् ।
यदागतौ लब्धिगुणौ विशोध्यौ स्वतक्षणाच्छेषमितौ तु तौ स्तः ॥252॥
[Ref. 1 (p.103)]
Purport: - (i) When number of partial quotients n is even, and addend c is positive the *remainders* are the least value of $y = y_0$ (लब्धि) and the least value of $x = x_0$ (गुणक).

(ii) When number of partial quotients n is odd and c is positive, or when n is even and c is negative, the above remainders are to be subtracted from the respective divisors to get their least values of $y = (a - y_0)$ (लब्धि), and $x = (b - x_0)$ (गुणक).

Example 1:

एकविंशतियुतं शतद्वयं यद्गुणं गणक पञ्चषष्टियुक् ।
पञ्चवर्जितशतद्वयोद्धृतं शुद्धिमेति गुणकं वदाऽऽशु तम् ॥252॥

[Ref. 1 (p.103)]
Purport: - O, mathematician, tell me the number when multiplied by 221 and increased by 65 is divisible by 195?

Answer: - The problem to solve is $y = \dfrac{221x + 65}{195}$.

According to step 1, HCF of *the dividend* 221 and *the divisor* 195 = 1is13, and it *divides the addend* 65. Therefore, $y = \dfrac{221x + 65}{195}$ can be solved for integral solutions of x and y.

According to step 2, when the *dividend* 221, *divisor* 195 and the *addend* 65 are divided by the HCF 13, the *reduced dividend* (17); *reduced divisor* (15) and *reduced addend* (5) are obtained. Then the reduced equation is; $y = \dfrac{17x + 5}{15}$.

According to step 3, the *reduced dividend* 17 and *reduced divisor* 15 are mutually divided (*as in the case of finding their* HCF*)* until the *remainder* 1 is obtained.

```
15)17(1
   15
   ---
    2)15(7
      14
      --
       1
```

This *Mutual division* gives the *quotients* 1 and 7. They are placed one below the other, and below them the *reduced addend* 5 of the equation and zero.

Apply the *Modified Vallika process* (Step 4) thus;

q_t	F_t	
1	$40 = F_2$	$F_{(n-t)} = q_t F_{(n-t-1)} + F_{(n-t-2)}$
7	$35 = F_1$	
5	$35 = F_{-1}$	
0	$0 = F_o$	

Modified Vallika process gives $F_1 = 35$ and $F_2 = 40$.

According to Step 5 (i);

$F_2 = 40 = 17 \text{ x } 2 + 6$, hence the least integral value of y is; $y_0 = 6$, and

$F_1 = 35 = 15 \text{ x } 2 + 5$, hence the least integral value of x is; $x_0 = 5$.

Number of partial quotients, n = 2 (*an even number*) and addend c is *positive*.

Therefore, $x = x_0 = 5$ and $y = y_0 = 6$ are the *least integral values of x* and y satisfy

$$y=\frac{17x+5}{15}=\frac{221x+65}{195}\ ,$$

Its *infinite integral solutions* for x and y are; $x =$ 15t + 6 and y = 17t + 5.

Any required values for them could be obtained by substituting a natural number to t.

Conclusion: -

<u>Joseph Louise Lagrange (1736-1813 AD)</u> found the least integral values of x and y for the problems like Ax + By = C (AD.1770) by dividing A and B mutually to find their HCF (k). Then got *reduced dividend* (a); reduced *divisor* (b); and *reduced addend* (c), by dividing A, B and C by the HCF (k) and got the reduced equation in the form $ax + by = c.$ Then Lagrange used the *partial quotients* of a *terminating continued fraction expansion* $\frac{a}{b}=[\mathrm{q}_1,\mathrm{q}_2,\mathrm{q}_3,\ ...\ ,\mathrm{q}_{n-1},\mathrm{q}_n]$ in *<u>a pair of recurring relations</u>*

$$\mathrm{q}_r x_{r-1}+x_{r-2}=x_r\,;\text{where } x_{-1}=1,\ x_0=0,$$

$$\mathrm{q}_r y_{r-1}+y_{r-2}=y_r\,;\text{where } y_{-1}=0,\ y_0=1,$$

In which r = (1, 2, 3 . . . (n-1), n)

The least integral values of x and y for the problem Ax + By = C are;

$x=x_n = \mathrm{q}_n x_{n-1}+x_{n-2}$ and $y=y_n = \mathrm{q}_n y_{n-1}+y_{n-2}$

<u>Bhaskara-II (born 1114 AD)</u> found the least integral values of x and y for the problems like $y=\left[\frac{\mathrm{A}x+\mathrm{C}}{\mathrm{B}}\right]$, by dividing A and B mutually to find their HCF (k). Then got *reduced dividend* (a); reduced *divisor* (b); and *reduced addend* (c), by diviing A, B and C by the HCF (k) and got the reduced equation in the form $y=\frac{ax+c}{b}=\left[\frac{\mathrm{A}x+\mathrm{C}}{\mathrm{B}}\right]$. Afterwards, a and b are mutually divided using a recurring operation;

$r_{(t-2)} = q_t\, r_{(t-1)} + r_t$; $0 \le r_t < r_{(t-1)}$, in which $r_{(-1)} = a$ and $r_0 = |b|$ for t = 1, 2, 3, . . . n. until $r_n = 1$ is obtained, the partial quotients q_1, q_2, q_3, . . . q_n . Bhaskara II used the partial quotients q_1, q_2, q_3, . . . q_n thus obtained in a *process* named *Vallika* (creeper like).

The *Vallika* (creeper like) could be represented by a *a single recurring relation*

$F_{(n-t+1)} = q_t F_{(n-t)} + F_{(n-t-1)}$ in which $F_0 = c$ and $F_{(-1)} = 0$ for [t = (n-1), (n-2), . . . , 2, 1] successively to get F_n and $F_{(n-1)}$.

The least integral values of y and x for the problems of the form $y = \frac{ax+c}{b} = \left[\frac{Ax+C}{B}\right]$ are found thus;

The least integral values of $y = y_0 =$ लब्धि, is found as the remainder while dividing F_n by the *reduced dividend* (a) in [$F_n = at + y_0$].

The least integral value of $x = x_0 =$ गुणक is found as the remainder by while dividing $F_{(n-1)}$ by the *reduced divisor* (b) in [$F_{(n-1)} = bt + x_0$].

Joseph Louise Lagrange (born 1736 AD) *gives a Method for solving Indeterminate equations in two unknowns for the least integral values using the partial quotients* of *the Continued Fraction Expansion of a rational number in a PAIR of recurring relations* (from top to bottom).

Bhaskara II (born 1114 AD) *gives a Method for solving Indeterminate equations in two unknowns for the least integral values using the partial quotients* of *the Continued Fraction Expansion of a rational number in a SINGLE recurring relation* (from bottom to top, *creeper like*).

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