Rashtriya Sanskrit Vidyapeetha, Tirupati, (Deemed University)
Directorate of Distance Education
Certificate Course on "*Scientific Heritage of India*"
Syllabus for Paper II

3. Mathematics.

Unit	Number of lessons	Lessons
3. 1	3. 1. 1 3. 1. 2	Numbers in Sanskrit Works; An Overview. Indian Decimal place-value system.
3. 2	 3. 2. 1 3. 2. 2 3. 2. 3	Numerals in Sanskrit Works; a) Words as Numerals b) Alphabets as Numerals and c) Early Magic Squares.
3. 3	3. 3. 1 3. 3. 2	a) *Aryabhatiya Numerals* b) Number of Revolutions of Geo-centric Planets in a Yuga (43,20,000 years) in *Aryabhatiya Numerals,* comparison of sidereal periods of Geo-centric planets in *Aryabhatiya* with their present-day values and Reason for naming the week-days.
3. 4	3. 4. 1	Glimpses of Mathematics of *Sulvakaras.*
3. 5	 **3. 5. 1** **3. 5. 2**	**Glimpses of Mathematics of *Aryabhata*-I** **1) *Arithmetic* and *Mensuration* in *Aryabhatiya.*** ***2) Circles* in *Aryabhatiya.***
3. 6	3. 6. 1 3. 6. 2	Glimpses of Mathematics of *Mahaveeracarya.*
3. 7	3. 7. 1 3. 7. 2	Glimpses of Mathematics of *Bhascara*-II.
3. 8	3. 8. 1 3. 8. 2	*Chandassutra; Zero* and Binary number System. Transmission of *Zero*, Decimal place-value system and *Indian Trigonometry* outside India Ratio of C*ircumference of a Circle to its Diameter* (π), in *Indian Mathematics*
3. 9	3. 9. 1 3. 9. 2	Biography of *Bharati Krishna Tirthaji* and Glimpses of His Contribution to *Indian Mathematics*
3. 10	3. 10. 1 3. 10. 2	Biography of *Srinivasa Ramanujan* and Glimpses of His Contribution to *Mathematics*
	20	**Total number of lessons**

Unit 3. 5 : Geometry in Aryabhatiya of *Aryabhata*-I

Structure: - The Unit: Glimpses of Mathematics of *Aryabhata*-I contains an introduction and two lessons.

Introduction gives information about the date of birth and the place where *Aryabhata*-I lived. It also gives a brief note about his famous work *Aryabhatiya*

First lesson; 8. 2. 1 *Arithmetic* and *Mensuration* has five parts;

3. 5. 1 (a): connects ideas of square and squaring of arithmetic operations with geometry.

3. 5. 1 (b): connects ideas of cube and cubing of arithmetic operations with geometry.

3. 5. 1 (c): gives a sloka and an explanation of Bhaskara-I about area of a triangle.

3. 5. 1 (d): gives a sloka on area of a trapezium.

3. 5. 1 (e): gives a sloka on theorem on square of hypotenuse

Second lesson; 3. 5. 2 C*ircles* has four parts.

3. 5. 2 (a): Area of a Circle.

3. 5. 2 (b): Ratio of circumference of a circle to its diameter.

3. 5. 2 (b): Theorem on square of half-chord of a Circle.

3. 5. 2 (d): To find arrows of intercepting arcs of intersecting circles.

Introduction: - *Aryabhatiya* is the composition of Aryabhata-I. Aryabhata-I lived at Kusumapura or Pataliputra in ancient Magadha, or modern Patna in Bihar. The year of birth of Aryabhata-I is known to us with precision through a verse in Aryabhatiya ;

षष्ट्यब्दानां षष्टिर्यदा व्यतीतास्त्र्ययश्च युगपादाः ।
त्र्यधिका विंशतिरब्दास्तदेह मम जन्मनोऽतीताः ॥

"*When sixty times sixty years and three quarter-yugas had elapsed* (of the current *yuga), twenty-three years had then passed since my birth.*"

This shows that in the kali year 3600 (elapsed), Aryabhata-I was *twenty-three years* of age. Since the kali year 3600 (elapsed), corresponds to A.D. 499, it follows that Aryabhata-I was born in the year A.D. 476. The Gupta king Buddhagupta reigned at Pataliputra from A.D. 476 (the year Aryabhata-I was born) to A.D. 496.

Aryabhatiya deals with both mathematics and astronomy. It contains 121 stanzas in all, and is marked for brevity and conciseness of composition. The subject matter of the *Aryabhatiya* is divided into 4 chapters, called *paada* (or section).

*Gitika paada, the f*irst *paada*, consists of 13 stanzas and explains basic definitions, important astronomical parameters and tables. It gives the definitions of the larger units of time (*Kalpa, Manu* and *yuga*), the circular units (sign, degree and minute) and the linear units (*yojana, nru, hasta* and *angula*). Revolutions of Geo-centric planets in a *yuga* (43,20,000 Years) and other astronomical data are given in his own invention *Aryabhatiya Numeral.*

Ganita paada, the second paada, consists of 33 stanzas and deals with mathematics. The topics dealt with are the geometrical figures, their properties, approximate value of *'the ratio of the circumference of a circle to its radius'* (approximate value of π), mensuration, problems on the shadows and gnomons, series, interest, simple equations, simultaneous equations, quadratic equations and linear indeterminate equations. The arithmetical method for extracting the square root and the cube root and rules meant for certain specific mathematical problems including the method of constructing the table *of Rsines.*

Kaalakriya paada, the third paada, consists of 25 stanzas and deals with Astronomical concepts after giving out the various units of time, divisions of the year, various kinds of year, lords of hours and days *etc.,*

Gola paada, the fourth paada, consists of 50 stanzas and deals with the motion of Geo-centric planets on the celestial sphere. It gives rules relating to the various problems of spherical astronomy. It also deals with the calculation and graphical representation of the eclipses and the visibility of the planets.

Lesson 3. 5. 1 : Arithmetic and Mensuration

3. 5. 1 (a): Square and squaring

Aryabhata-I defines the area of a square figure in geometry and extends the definition of the term 'square' as the product of two equal quantities.

वर्गः समचतुरश्रः फलं च सदृशद्वयस्य संवर्गः ।

Bhaskara-I, in his *Aryabhatiya Bhasya* (629 A.D.) explains the above *sloka* thus;

वर्गः करणी कृतिः वर्गणा यावक्करणमिति पर्यायाः । समचतुरश्रक्षेत्रविविशेशः संञ्ज्ञी, वर्गः संञ्ज्ञा । अत्र संञ्ज्ञिसंञ्ज्ञयोरभेदेन उच्यते 'वर्गः समचतुरश्रः' इति । अस्माद् यो यो वर्गः समचतुरश्रक्षेत्रविशेषः । एवं फलं च सदृशद्वयस्य संवर्गः । संवर्गो घातो गुणना हतिरुद्धर्तना इति पर्यायाः ।

An equilateral quadrilateral with equal diagonals (a square figure) *and area thereof are called 'square'. The product of two equal quantities is also called 'square'.*

Bhaskara-I gives the terms वर्गः, करणी, कृतिः, वर्गणा and यावक्करण as synonyms, meaning 'square or squaring'. Of these करणी, वर्गणा and यावक्करण are unusual. The term यावक्करण is derived from the fact that in Hindu algebra x^2 is written as याव (या) for यावत्–तावत् , *i.e.* x and व for वर्ग, *i.e.* square.

3. 5. 1 (b): Cube and Cubing

Aryabhata-I defines the volume of a figure 'cube' in geometry and extends the definition of the term 'cube' as the product of three equal quantities.

सदृशत्रयसंवर्गो घनस्तथा द्वादशाश्रिः स्यात् ॥३॥

Bhaskara-I, in his *Aryabhatiya Bhasya* (629 A.D.) explains the above *sloka* thus;

सदृशत्रयसंवर्गः । सदृशत्रयसंवर्गो घनो भवति । घनो वृन्दं सदृशत्रयाभ्यास इति पर्यायाः । स च द्वादशाश्रिः द्वादश अश्रयो यस्य सोऽयं द्वादशाश्रिः, स्यात् भवेत् । 'तथा'शब्देन समचतुरश्रतां घनस्य प्रतिपादयति । अन्तरेणापि 'तथा'शब्दं अस्य घनस्य समचतुरश्रता शक्यत एव प्रतिपत्तुम् । 'वर्गः समचतुरश्रः' इत्यत्राधिकृतं समचतुरश्रग्रहणमनुवर्तते, अश्रयो यस्य मृदान्येन वा प्रदर्शितव्याः ।

The continued product of three equals as also the cuboidal solid having twelve equal edges is called a 'cube'.

3. 5. 1 (c): Area of a triangle

Aryabhata-I gives the formula to find the area of a scalene triangle when its base and altitude are known.;

त्रिभुजक्षेत्रफलम् –

त्रिभुजस्य फलशरीरं समदलकौटीभुजार्धसंवर्गः ।

The product of the perpendicular (*dropped from the vertex on the base*) *and half the base gives the measure of the area of a triangle.*

Bhaskara-I, in his *Aryabhatiya Bhasya* (629 A.D.) explains the term समदलकौटी.

The term समदलकौटी means '*the perpendicular dropped from the vertex on the base of a triangle*' *i.e,.* 'the altitude of a triangle'. Bhaskara-I criticized those who interpreted it as 'the upright that bisects the triangle into two equal segments', for, in that case, the above rule will be applicable only to equilateral and isosceles triangles.

Explanation of Bhaskara-I, in detail, is;

तिस्रौ भुजा यस्य क्षैत्रस्य तदिदं क्षैत्रं त्रिभुजम् । भुजा बाहुः पार्श्वमिति पर्यायाः । तत्र त्रीणि क्षैत्राणि सम–द्विसम–विषमाणि ! त्रिभुजस्यैति त्रिभुजक्षेत्रजातिमङ्गीकृत्यैकवचननिर्दैशः । तस्य त्रिभुजस्य फलशरीरम् । फलस्य शरीरं फलशरीरं, फलप्रमाणमित्यर्थः । समदलकौटीभुजार्धसंवर्गः ।

समदलकौटी, अवलम्बकः । अत्र केचित् – समे दले यस्याः सैयं समदला, समदला चासौ ककौटी च समदलकौटीति वर्णयन्ति । तेषां सम–द्विसम–त्र्यश्रक्षेत्रयौरेव फलसिद्धिः, न विषमत्र्यश्रक्षैत्रस्य । अस्माकं पुनः समदलकौटीत्यनेनावलम्बकव्युत्पत्या ब्रुवतां त्रयाणामपि फलानयनं सिद्धम् । अथवा यै व्युत्पत्तिं कुर्वन्ति तेषामपि त्रयाणां त्र्यश्रक्षैत्राणां फलानयनं सिद्धमेव ! कुतः ? रूढैषु क्रिया व्युत्पत्तिकर्मार्था नार्थक्रिया इति ! भुजाया अर्धं भुजार्धम् । अथात्र भुजाशब्देन भुजा बाहुः पार्श्वमिति सामान्येन त्रयाणां पार्श्वानां प्रतिपत्तौ प्रसक्तायां विशिष्टा एव भुजा परिगृह्यते, भुजासंज्ञिता ! सामान्यचोदनाश्च विशेषेऽवतिष्ठन्त इति । अत्र गणितै भुजाशब्दः औणादिकः प्रतिपत्तव्यः अन्यथा हि भुजान्युब्जौ पाण्युपतापयौः [अष्टाध्यायी, ७. ३. ६९] इति भुजाशब्दस्य पाणावर्थे निपातितत्त्वात् क्षेत्रपार्श्वे न लभ्यते । तस्याभुजाया अर्धं भुजार्धम् ।

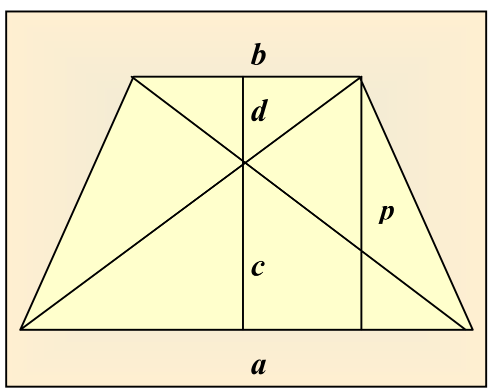
समदलकौट्या भुजार्धस्य च–संवर्गः समदलकोटीभुजार्धसंवर्गः, त्रिभुजस्य फलशरीरं भवति ।

Therefore,
Area of a triangle of the known base and altitude = (1/ 2) (base) (altitude)

3. 5. 1 (d): Area of a trapezium
Aryabhata-I gives the formula to find the length of perpendiculars drawn from the point of intersection of its diagonals to its opposite parallel sides and area of the trapezium when the lengths of its parallel sides and the distance between them are known.

आयामगुणे पार्श्वे तद्योगहृते स्वपातरेखे ते । विस्तरयोगार्धगुणे ज्ञेयं क्षेत्रफलमायामे ॥८॥

(*Severally*) *multiply the base and the face* (of the trapezium) *by the height, and divide* (each product) *by the sum of the base and the face : the results are the lengths of the perpendiculars on the base and the face* (from the point of intersection of the diagonals). *The results obtained by multiplying half the sum of the base and the face by the height is to be known as the area* (of the trapezium).



Let *a, b* be the base and the face, *p* the height and *c, d* are the lengths of the perpendiculars on the base and the face from the point of intersection of the diagonals. Then

$$c = \frac{ap}{a+b} \quad \text{and} \quad d = \frac{bp}{a+b}.$$

Area of the trapezium $= \frac{1}{2}(a+b)p$

3. 5. 1 (e): Theorem on square of hypotenuse

Aryabhata-I gives the formula thus;

यश्चैव भुजावर्गः कोटीवर्गश्च कर्णवर्गः सः ।

(*In a right angled triangle*) *the square of the base plus the square of the upright is the square of the hypotenuse.*

यश्च भुजाया वर्गः, यश्च कोटिवर्गः, तयोर्योगः कर्णवर्गः । (तस्य मूलं कर्णो भवति) ।

This theorem is the so-called '*Pythagoras theorem'*, which was known to *Sulvakaras* (800 B.C.). The theorem is to be renamed as *Baudhayana theorem* or '*Theorem of square on the diagonal of a rectangle'.*

Lesson 3. 5. 2: Circles

3. 5. 2 (a): Area of a Circle.

Aryabhata-I gives the formula to find the area of a circle ;

समपरिणाहस्यार्धं विष्कम्भर्धहतमेव वृत्तफलम् ।

Half of the circumference, multiplied by the semi-diameter certainly gives the area of a circle.

Bhaskara-I, in his *Aryabhatiya Bhasya* (629 A.D.) explains the meaning of the sloka thus;

परिणाहः परिधिः । समश्चासौ परिणाहश्च समपरिणाहः, तस्यार्धम् । विष्कम्भो व्यासः, तस्यार्धं विष्कम्भार्धं, तेन हतं विष्कम्भर्धहतम्, विष्कम्भर्धगुणितमिति यावत् । समपरिणाहस्यार्धं विष्कम्भर्धहतमेव वृत्तफलम् ।

$$\text{Area of a circle} = \left(\frac{\text{circumference}}{2}\right)\left(\frac{\text{diameter}}{2}\right) = \left(\frac{2\pi r}{2}\right)\left(\frac{2r}{2}\right) = \pi r^2$$

Thus, it could be seen that Aryabhata's formula for area of a circle is different from the popular formula, in form. He has used the term 'half the diameter' instead of 'radius', wherever necessary.

3. 5. 2 (b): Ratio of circumference of a circle to its diameter.

Aryabhata-I gives an approximate value of the *Ratio of circumference of a circle to its diameter,* correct to four decimal places.

The sloka reads thus ;

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम् ।
अयुतद्वयविष्कम्भस्यासन्नो वृत्तपरिणाहः ॥१०॥

Hundred plus four, multiplied bu eight, and added to sixty-two thousand ; this is the nearly approximate measure of the circumference of a circle whose diameter is twenty thousand.

Bhaskara-I, in his *Aryabhatiya Bhasya* (629 A.D.) explains the meaning of the sloka thus;

चतुर्भिरधिकं चतुरधिकम् । किं तत् ? शतम् । अष्टाभिर्गुणितम् अष्टगुणम् । एतदुक्तं भवति–अष्टौ शतानि द्वात्रिंशदुत्तराणीति । सहस्राणि च द्वाषष्टिः । एतदुभयमेकत्र ६२८३२ । अयुतद्वयं च विष्कम्भश्च अयुतद्वयविष्कम्भः । अथवा अयुतद्वयसङ्ख्यो विष्कम्भोऽयुतद्वय–प्रमाणो वा अयुतद्वयविष्कम्भः । तस्य अयुतद्वयविष्कम्भस्य । स च २०००० । आसन्नः निकटः । सूक्ष्मस्य परिणाहस्य । आसन्न शब्देन तत्स्मीपवर्तिनाभिधीयते ।

This gives ;

$$(100 + 4)\ 8 + 62{,}000 = 62{,}832 = \text{Circumference}$$

$$\pi \times \text{diameter} = \text{Circumference}$$

$$\pi = \frac{62832}{20000} = 3.1416$$

$$\pi \times 20{,}000 = 62{,}832$$

This value does not occur in any earlier work on mathematics, and forms an important contribution of Aryabhata-I. This value could be obtained by finding the perimeters of inscribed regular polygons, starting with hexagon and successively doubling the number of sides of these inscribed polygon. Aryabhata's approximate value is obtained by considering the perimeter of an inscribed regular polygon of 768 sides(which could be obtained using a scientific calculator). Jaina mathematicians used $\sqrt{10}$ as an approximation to the ratio of circumference of a circle to its diameter (π) by considering

$$\pi = \frac{62832}{20000} = 3.1416$$

the perimeter of an inscribed regular polygon of 12 sides (dodecagon).

It is noteworthy that Aryabhata-I has called the value for π approximate (आसन्नः, निकटः).

The present day value of π = 3.141592654… Is it not surprising to note Aryabhata's (b. A.D. 476) value of π being approximated to its fourth decimal place during such a distance past, without any devise to work out with huge numbers? Or did they have any devise?

3. 5. 2 (c): Theorem on square of half-chord of a Circle.

Aryabhata-I gives the rule thus;

वृत्ते शरसंवर्गोऽर्धज्यावर्गः य खलु धनुषो ॥१७॥

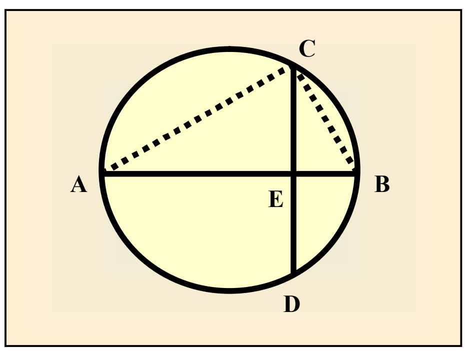
In a circle (when a chord divides it into two arcs), *the product of the arrows of the two arcs is certainly equal to the square of half the chord.*

Bhaskara-I, in his *Aryabhatiya Bhasya* (629 A.D.) explains the meaning of the sloka thus;

वृत्ते क्षेत्रे, शरयोः संवर्गः शरसंवर्गः, सः अर्धज्यावर्गो भवति। ।

स खलु धनुषोः, तयोरेव धनुषोरर्धज्यावर्गो भवति ।

In a circle, a chord CD and a diameter AB intersect at right angles at E.



Arc DBC is called *dhanu* of chord CD (*Jya*), and Arc DAC is another *dhanu* of chord CD (*Jya*).

EA is *shara* of *dhanu* DBC and EB is *shara* of *dhanu* DAC.

Then according to the theorem ;

(Shara EA of *dhanu* DBC) x (Shara EB of *dhanu* DAC) = (half of chord CD)^2

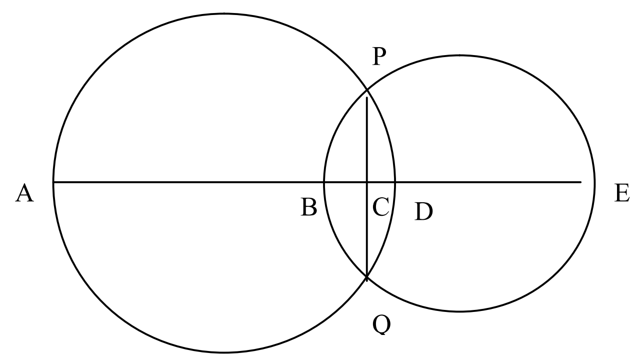
i.e., AE x EB = CE^2

This result is proved through the property of similar triangles of triangle AEC and triangle CEB in Euclidean geometry. Since Indian mathematicians were practical mathematicians, he did not record the proof for their results.

3. 5. 2 (d): To find arrows of intercepting arcs of intersecting circles:
Aryabhata-I gives the rule thus;
ग्रासेन द्वे वृत्ते ग्रासगुणे भाजयेत् पृथक्त्वेन । ग्रासेनयोगलब्ध्यौ सम्पातशरौ परस्परतः ॥१८॥

(*When one circle intersects another circle) multiply the diameters of the two circles each diminished by the erosion, by the erosion and divide (each result) by the sum of the two circles after each has been diminished by the erosion : then the arrows of the arcs (of the two circles) intercepted in each other are obtained*.)

Let two circles intersect at P and Q, and ABCDE be the line passing through the centers of the two circles. Then BD is the erosion (ग्रास), and BC and CD are the arrows of the intercepted arcs. [The arrow (उत्क्रमज्या, R*versine*) of a circle is one of the three basic trigonometric functions;
R*sine* (ज्या), R*cosine* (कोटिज्या) and R*versine* (उत्क्रमज्या).]



The Rule states that

$$BC = \frac{(AD - BD).BD}{(AD - BD) + (BE - BD)} \quad \text{and} \quad CD = \frac{(BE - BD).BD}{(AD - BD) + (BE - BD)}$$

Proof: - As per the 'Theorem on square of half-chord of a Circle' [8. 2. 2 (c)]

$$AC \times CD = BC \times CE = CE^2$$

Therefore, $(AD - BD + BC)\ (BD - BC) = BC\ (BE - BC)$ also,

$$(AD - CD)\,CD = (BD - CD)\,(BE - BD + CD)$$

from, $$(AD - BD + BC)\,(BD - BC) = BC\,(BE - BC)$$

$$(AD - BD + BC)\,BD - BC\,(AD - BD + BC) = BC\,(BE - BC)$$

$$(AD - BD)\,BD = -\,BC.\,BD + BC\,(AD - BD + BC) + BC\,(BE - BC)$$

$$BC\,[(-\,BD + AD - BD + BC + BE - BC)] = (AD - BD)\,BD$$

$$BC\,[(AD - BD) + (BE - BD)] = (AD - BD)\,BD$$

$$BC = \frac{(AD - BD).\,BD}{(AD - BD) + (BE - BD)}$$

Similarly, from $(AD - CD)\,CD = (BD - CD)\,(BE - BD + CD)$ the rule;

$$CD = \frac{(BE - BD).\,BD}{(AD - BD) + (BE - BD)}$$ could be proved.

Exercise: -

1. How does Aryabhata connect the terms 'square and squaring' and 'cube and cubing' between geometry and arithmetic?
2. What is the objection of Bhaskara-I about the interpretation of the word समदलकौटी
   by the earlier commentators?
3. State Aryabhata theorem on square of hypotenuse.
4. State the rule of *Aryabhata*-I for finding the area of a circle.
5. What might be the basis for Indian mathematicians to find the ratio of circumference of a circle to its diameter?
6. How could *Jaina mathematicians* arrive at the circumference of a circle of unit diameter and what is its value?
7. What are the basic trigonometric functions in Indian mathematics?
8. What are the formulae to find R*versine* (उत्क्रमज्या)
9. in terms of segments formed on combined diameter of two intersecting circles and the perpendicular to it joining the points of intersection of the two circles?